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On non-commutative G_2 structure

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Abstract

Using an algebraic orbifold method, we present non-commutative aspects of G_2 structure of seven-dimensional real manifolds. We first develop and solve the non-commutativity parameter constraint equations defining G_2 manifold algebras. We show that there are eight possible solutions for this extended structure, one of which corresponds to the commutative case. Then, we obtain a matrix representation solving such algebras using combinatorial arguments. An application to matrix model of M-theory is discussed.

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1. Introduction

It has been known for a long time that non-commutative (NC) geometry plays an interesting role in the context of string theory [1] and, more recently, in certain compactifications of the matrix formulation of M-theory on NC tori [2]. These studies have opened new lines of research devoted, for example, to the study of supermembrane and Hamiltonian [3–9].

In the context of string theory [10], NC geometry appears from the study of the quantum properties of D-branes in the presence of nonzero B-field. In particular, for large values of B-field, the spacetime worldvolume coordinates no longer commute and the usual product of functions is replaced by the star product of Moyal bracket. In the context of M-theory, NC structure is also present [7, 8]. For instance, it can emerge from the L.C.G. Hamiltonian of the supermembrane with fixed central charge. The central charge can be induced by an irreducible winding around a flat even torus [6, 8, 9]. Using minimal immersions associated with a symplectic matrix of central charges as backgrounds, a non-commutative symplectic supersymmetric Yang–Mills theory, coupled to the scalar fields transverse to the supermembrane, has been obtained. The main physical interest of this NC formulation of the supermembrane relies on the discreteness of its quantum spectrum [4] and mainly [5] in clear contrast to the commutative case [11, 12]. NC geometry has also been used to study tachyon condensation [13]. However, most of the NC spaces considered in all these studies involve mainly NC $\mathbf{R}^d_\theta$ [13] and NC tori T^d_θ [14].

Recently efforts have been devoted to go beyond these particular geometries by considering NC Calabi–Yau manifolds used in string theory compactifications. Great interest has been given to building NC Calabi–Yau (NCCY) threefolds using the so-called algebraic geometry method introduced first by Berenstein and Leigh in [15]. The NC aspect of this manifold is quite important for the stringy resolution of Calabi–Yau singularities and the interpretation of string states. In particular, in string theory on Calabi–Yau manifolds, NC deformation is associated with open string states while the commutative geometry is in one-to-one correspondence with closed string states. This formulation has been extended to higher-dimensional manifolds being understood as homogeneous hypersurfaces in $\mathbf{CP}^{n+1}$ [16] or more generally to hypersurfaces in toric varieties [17, 18].

The aim of this paper is to extend those results to the case of manifolds with exceptional holonomy groups used in string theory compactifications. In particular, our main objective is to develop an explicit analysis for a NC G_2 structure. Using an algebraic orbifold method, we present NC aspects of manifolds with G_2 holonomy group. We develop and solve the non-commutativity parameter constraint equations defining such structure algebras. We find that there are eight possible solutions for this extended structure, one of which corresponds to the commutative case. Using combinatorial arguments, we give matrix representations solving such algebras. A matrix model of M-theory on G_2 manifolds is found. Its generalization to NC G_2 is discussed.

The outline of the paper is as follows. In section 2, we present NC aspects of G_2 structure manifolds using an algebraic orbifold method. In section 3, we develop and solve the parameter constraint equations defining such a structure. Then, we show that there are eight possible solutions for this extended geometry. In section 4, we give matrix representations solving such algebras. An application to matrix model of M-theory is discussed in section 5. Then, we give our conclusion in section 6. We finish this work with an appendix.

2. NC G_2 structure

In this section, we want to present a non-commutative geometry with the G_2 holonomy group. This may extend results of the NC Calabi–Yau geometries. First, the G_2 structure appears in a seven real dimensional manifold, with holonomy group G_2, and plays a crucial role in the M-theory compactification. In particular, it was shown that in order to get four-dimensional models with only four supercharges from M-theory, it is necessary to consider a compactification on such a structure manifold. As in the Calabi–Yau case, there are several realizations and many non-trivial $N = 1$ models in four dimensions could be derived, from M-theory, once a geometric realization has been considered. Before going ahead let us recall: what is the commutative G_2 structure? Indeed, consider a $\mathbf{R}^7$ parametrized by $(y_1, y_2, \ldots, y_7)$. On this space, one defines the metric

$$g = \mathrm{d}y_1^2 + \cdots + \mathrm{d}y_7^2. \tag{1}$$

Reducing $SO(7)$ to G_2, one can also define a special real three-form as follows:

$$\varphi = \mathrm{d}y_{123} + \mathrm{d}y_{145} + \mathrm{d}y_{167} + \mathrm{d}y_{246} - \mathrm{d}y_{257} - \mathrm{d}y_{356} - \mathrm{d}y_{347} \tag{2}$$

where $\mathrm{d}y_{ijk}$ denotes $\mathrm{d}y_i\,\mathrm{d}y_j\,\mathrm{d}y_k$. This expression of φ comes from the fact that G_2 acts as an automorphism group on the octonion algebra structure given by

$$t_i t_j = -\delta_{ij} + f_{ij}^k t_k, \tag{3}$$

which yields the correspondence

$$f_{ij}^{k} \to \mathrm{d}y_{ijk}. \tag{4}$$

The couple (g, φ) defines the so-called G_2 structure.

In what follows, we want to deform the above structure by introducing NC geometry. This deformation may extend results of NC Calabi–Yau geometries studied in [15–20]. It could also be used to resolve the G_2 manifold singularities by non-commutative algebraic method.

Simply speaking, the G_2 structure could be deformed by imposing the constraint

$$y_i y_j \neq y_j y_i. \tag{5}$$

Basically, there are several ways to approach such a deformed geometry. For instance, one may use the string theory approach developed by Seiberg and Witten in [10]. Another way, in which we are interested in this present work, is to use an algebraic geometry method based on solving the non-commutativity in terms of discrete isometries of orbifolds [15, 16].

2.1. Constraint equations of NC G_2 structure

To get the constraint equations defining NC G_2 structure, we proceed in steps as follows. First, we consider a discrete symmetry Γ, which will be specified later on, acting as follows:

$$\Gamma : \quad y_i \to \alpha_i y_i, \qquad \alpha_i \in \Gamma. \tag{6}$$

The resulting space is constructed by identifying the points which are in the same orbit under the action of the group, i.e., $y_i \to \alpha_i y_i$. It is smooth everywhere, except at the fixed points, which are invariant under non-trivial group elements of Γ. The invariance of the G_2 structure under Γ can define a non-compact seven-dimensional manifold with holonomy group G_2. The compactification of this geometry leads to models studied by Joyce in [21].[3] Then, we see the orbifold space as a NC algebra. We seek to deform the algebra of functions on the orbifold of $\mathbf{R^7}$ to a NC algebra $\mathcal{A}_{\mathrm{nc}}$. In this way, the centre of this algebra is generated by the quantities invariant under the orbifold symmetry. In particular, we imitate the BL orbifold method given in [15] to build a NC extension of $(\mathbf{R^7}/\Gamma)_{\mathrm{nc}}$. This extension is obtained, as usual, by extending the commutative algebra $\mathcal{A}_c$ of functions on $\mathbf{R^7}$ to a NC one $\mathcal{A}_{\mathrm{nc}} \sim (\mathbf{R}^7/\Gamma)_{\mathrm{nc}}$. The NC version of the orbifold $\mathbf{R^7}/\Gamma$ is obtained by substituting the usual commuting y_i by the matrix operators Y_i satisfying the following NC algebra structure[4]:

$$Y_i Y_j = \Theta_{ij} Y_j Y_i, \tag{7}$$

where Θ is a matrix with further properties arranged in such a way as to preserve the G_2 structure. In this way, Θ_{ij} should satisfy some constraint equations defining the explicit NC G_2 structure. Here, we want to derive such parameter constraint relations. To do so, let us start by writing down the trivial ones. Indeed, equation (7) requires that

$$\Theta_{ij}\Theta_{ji} = \Theta_{ii} = 1. \tag{8}$$

However, non-trivial relations come from the structure defining the commutative geometry. The crucial property in our method is that the entries of the Θ matrix must belong to Γ, i.e.,

$$\Theta_{ij} \in \Gamma. \tag{9}$$

The invariance of the G_2 structure under Γ requires that

$$\Theta_{ij}\Theta_{ik} = 1 \qquad \text{for} \qquad i \neq j \neq k, \qquad i, j, k = 1, \ldots, 7 \tag{10}$$

$$\Theta_{ij}\Theta_{ik}\Theta_{i\ell} = 1 \qquad \text{for} \qquad i \neq j \neq k \neq \ell, \qquad i, j, k, l = 1, \ldots, 7. \tag{11}$$

Let us recall that Θ_{ij} are coefficients of the matrix Θ, so no summation on the indices is considered.

[3] Other realizations of G_2 manifolds have been developed mainly in the context of M-theory compactifications.

[4] This algebra can be viewed as the Yang–Baxter equations.

2.2. Solving the constraint equations

Before studying the corresponding matrix representation, we first solve the above parameter constraint equations. This is needed to define explicitly the NC G_2 structure. It turns out that an explicit solution can be obtained once we know the elements of the centre $\mathcal{Z}(\mathcal{A}_{\rm nc})$. The latter, which yields the commutative algebra generated by quantities invariant under the action of Γ, is just the commutative G_2 geometry. It may be a singular manifold while the geometry corresponding to the NC algebra will be a deformed one. In other words, the commutative singularity can be deformed in a NC version of orbifolds and can have a physical interpretation in M-theory compactifications.

Let us now specify the discrete group Γ. In order not to loose contact with the commutative case, consider Γ as $Z_2 \times Z_2 \times Z_2$.[5] This symmetry has been studied in [21] to construct compact G_2 manifolds. Since in this case, y_i^2 is invariant under Γ, then the corresponding operator Y_i^2 should be at the centre of the $\mathcal{Z}(\mathcal{A}_{\rm nc})$. This implies that

$$\Theta_{ij}\Theta_{ij} = 1, \tag{12}$$

which is consistent with the invariance of the metric. This equation is a strong constraint which will have a serious consequence on solving NC G_2 structure. Equation (12) can be solved by taking Θ_{ij} as

$$\Theta_{ij} = (-1)^{\epsilon_{ij}}. \tag{13}$$

Here ϵ_{ij} is a matrix such that $\epsilon_{ij} + \epsilon_{ji}$ is equal to zero modulo 2, which is required by (8)–(10). A possible solution is given by $\Theta_{ij} = (-1)$ where $\epsilon_{ij} = 1$ modulo 2, corresponding to a flat space. However, the invariance of the G_2 structure leads to a solution where some Θ_{ij} are equal to 1. Using (7)–(13), one can solve Θ_{ij} as follows:

$$\Theta_{ij} = \begin{pmatrix} 1 & a & a & b & b & c & c \\ a & 1 & a & d & e & d & e \\ a & a & 1 & f & g & g & f \\ b & d & f & 1 & b & d & f \\ b & e & g & b & 1 & g & e \\ c & d & g & d & g & 1 & c \\ c & e & f & f & e & c & 1 \end{pmatrix}, \tag{14}$$

where the entries of this matrix are integers such that

$$a, b, c, d, e, f, g = \pm 1 \tag{15}$$

$$a = bdg \tag{16}$$

$$a = bc = de = gf. \tag{17}$$

Since the representation of the above algebra depends on this matrix, let us make two comments. First, the algebra of NC G_2 structure contains commutation and anticommutation relations. Second, we find there are eight different solutions corresponding to the two different choices of one of the integers, namely, $a = \pm 1$. They are classified as follows:

[5] This could be extended to any discrete subgroup of G_2 Lie group preserving the G_2 structure.

a	b	c	d	e	f	g
1	1	1	1	1	1	1
1	−1	−1	−1	−1	1	1
1	−1	−1	1	1	−1	−1
1	1	1	−1	−1	−1	−1
−1	−1	1	−1	1	1	−1
−1	1	−1	1	−1	1	−1
−1	1	−1	−1	1	−1	1
−1	−1	1	1	−1	−1	1

. (18)

From this classification, one can learn that we have eight different representations solving the NC G_2 structure. The trivial one corresponds to all the parameters being equal to +1, which is equivalent to having a complete set of commutative relations as a subset of possible solutions.

3. Matrix representation of NC G_2 structure

In this section, we construct eight different representations $\{Y_i\}_{a,b,c,\ldots,g}$ corresponding to the above NC G_2 structure. Our representation will be given in terms of infinite-dimensional matrices with the following block structure:

$$Y_i = \begin{pmatrix} M_i & 0 & 0 & 0 & \cdots \\ 0 & M_i & 0 & 0 & \cdots \\ 0 & 0 & M_i & 0 & \cdots \\ 0 & 0 & 0 & M_i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad i = 1, \ldots, 7. \tag{19}$$

Here M_i, which are $2^7 \times 2^7$ matrices, satisfy the NC G_2 structure given by (7).[6] The constraint $\Theta_{ii} = 1$ require that M_i should be symmetric matrices (i.e., $M_{mn} = M_{nm}$).

Our way to give explicit representations is based on the matrix realization of the Grassmannian algebra of spinors $SO(7)$ in 11 dimensions found in [5] although there are some differences in our case. The entries of the matrices M_i are $\{+1, -1, 0\}$. The vanishing coefficients are the same as in the symmetrized version of the matrix representation found in [5]. However, the nonvanishing entries differ in their signs with respect to [5]. For each M_i, the signs are determined by the NC G_2 structure, through Θ_{ij}, in the following way. Indeed, let us define a vector s_i that we shall call a vector of signs as

$$s_i = (+, \Theta_{1i}) \otimes \ldots \otimes (+, \Theta_{(i-1)i}), \qquad i = 1, \ldots, 7 \tag{20}$$

with this product defined as

$$(a_1, \ldots, a_k) \otimes (b_1, b_2) \equiv (a_1 b_1, \ldots, a_k b_1, a_1 b_2, \ldots, a_k b_2). \tag{21}$$

The kth element of this vector has the following expression:

$$(s_i)_k = \begin{cases} \displaystyle\prod_{\ell=1}^{i-1} \Theta_{\ell i} & 2^{\ell-1} + 1 + 2^{\ell} p \leqslant k \leqslant 2^{\ell}(p+1) \quad p = 0, \ldots, 2^{i-(\ell+1)} - 1 \\ + & \text{otherwise.} \end{cases} \tag{22}$$

In terms of this $(s_i)_k$, the matrices M_i can be expressed as

$$(M_i)_{mn} = \begin{cases} (s_i)_k & m = k + 2^i v \qquad n = m + 2^{i-1} \\ & k = 1, \ldots, 2^{i-1} \quad v = 0, \ldots, 2^{7-i} - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Note that $(s_i)_k$ for a given matrix M_i is repeated over v.

[6] Note that the size is related to the number of generators. Details are given in [5].

It is easy to see that M_i are hermitic traceless matrices of $2^7 \times 2^7$ size that satisfy the NC G_2 algebra. In order to extend our results to NC $T^7/(Z_2^3)_\Theta$, we can construct eight infinite-dimensional representations Y_i using (19) by considering that the periodic boundary condition is imposed at the ∞. Furthermore, our representation could solve a general algebra satisfying

$$U_i V_j = \Theta_{ij} V_j U_i \tag{24}$$

where Θ is a matrix containing real roots of the identity.

4. Link to matrix model of M-theory

In this section, we want to study an application of NC G_2 structure in M-theory compactifications. In particular, we discuss a possible link to matrix model of M-theory. Before going ahead, let us first review such a formalism and then try to connect it to our NC G_2 structure solutions. Indeed, matrix model of M-theory is defined by maximally supersymmetric $U(N)$ gauge quantum mechanics [23]. In the infinite momentum frame, this dynamic is described by the following SYM Lagrangian:

$$S_{\mathrm{D0}} = \frac{-1}{2gl_s}\,\mathrm{tr}((\dot{X}_i)^2 + \frac{1}{2}[X_i, X_j]^2 + \Psi^t(\mathrm{i}\dot{\Psi} - \Gamma^i[X_i, \Psi]). \tag{25}$$

Here X_i are nine Hermitian $N \times N$ matrices representing the transverse coordinates to N D0-branes in type IIA superstring theory. For no commuting transverse coordinates, this leads to fuzzy geometry in M-theory compactifications.

In what follows, we want to interpret seven of these X_i as operators satisfying our NC G_2 structure, while we take $X_8, X_9 = 0$. In this way, the vacuum equations of motion, for the static solutions, read

$$\sum_i [X_i, [X_i, X_j]] = 2\sum_i (1 - \Theta_{ij}) X_j, \tag{26}$$

where $X_i^{2N} = \mathbf{I}_{N\times N}$.

A simple computation reveals that the commutative solution of G_2 structure immediately solves (26). In this case, the X_i matrices can be diagonalized and their N eigenvalues represent the positions of the N D0-branes. However, the NC solutions do not satisfy (26). It is easy to see by using the result given in (18). Indeed, for each case, one gets

$$\sum_i [X_i, [X_i, X_j]] = 2\sum_i (1 - \Theta_{ij}) X_j = \begin{cases} \neq 0 & i \neq j \\ 0 & i = j. \end{cases} \tag{27}$$

However, one can solve the equations of motion even for the NC solutions by using a physical modification. Before doing that, let us first make a comment. We note that a singular characteristic of NC G_2 structure solutions corresponds to the case where one coordinate, which represents a direction in the transverse space, commutes with the remaining ones.[7]

[7] Our configuration for the non-commutative cases $(R^7/Z_2^3)_\Theta$ is acting as a $(R^6/Z_2^3)_{\widetilde{\Theta}} \times R$ where

$$\Theta_{ij} = \begin{pmatrix} 1 & \dots & \cdots & 1 \\ \vdots & & & \\ & & \widetilde{\Theta}_{i'j'} & \\ \vdots & & & \\ 1 & & & \end{pmatrix}, \qquad i, j = 1, \dots, 7, \qquad i', j' = 2, \dots, 7. \tag{28}$$

For a solution with $[X_1, X_{i'}] = 0$ with X_1 being coordinate of R and the rest of the solutions satisfying the algebra $X_{i'}X_{j'} = \widetilde{\Theta}_{i'j'}X_{j'}X_{i'}$. This is a general structure for all of our NC solutions.

In this way, the seven different solutions represent the different possible choices of this direction. The commutative coordinate represents a flat direction in the potential. It turns out that NC solutions could solve equations (26) if one introduces higher energy corrections in the D0-brane action. This generates a constant quadratic contribution in the action which now takes the following form:

$$S = S_{\mathrm{D0}} + \mu_j A_j^2. \tag{29}$$

Setting $\mu_j = \frac{1}{4}\sum_i(\Theta_{ii} - \Theta_{ij})$, (26) are now satisfied. Indeed, these mass terms are directly induced by anticommutative contributions of NC G_2 structure providing a massive potential for six of the seven directions. For a given flat direction, let say X_1, we have

$$\mu_j = \begin{cases} 0 & j = 1 \\ 8 & j = 2, \ldots, 7. \end{cases} \tag{30}$$

To find a link with the matrix model of M-theory for NC solutions, we should find explicit terms for couplings leading to mass terms in the above action, being related to NC G_2 structure in the regularized models. Alternatively, mass terms have been present in matrix models which are a regularization of theories also containing more contributions in the action as cubic or quartic terms. This is a common fact when fluxes are turned on in a theory as happens in Myers effect [24] or in soft breaking terms [25].

On the other hand, the Hamiltonian of the supermembrane matrix model with non-trivial central charge on a two torus has been studied in [3–5]. It is a non-commutative symplectic super Yang–Mills coupled to transverse scalar fields of the supermembrane. It contains quadratic, cubic and quartic contributions. However, if we restrict ourselves to the bosonic sector and fix the gauge field $A_r = 0$, the regularized model is reduced to

$$S_{\mathrm{D0}} = \frac{-1}{g^2}\,\mathrm{tr}\left(\frac{1}{4}[X^m, X^n]^2 + (\widehat{\lambda}_r X^m)^2\right) \qquad n, m = 1, \ldots, 7 \tag{31}$$

$$\widehat{\lambda}_{r\,B}^{\;A} = f^A_{r(B-r)} \qquad r = (1, 0), (0, 1). \tag{32}$$

This is the matrix model expansion of

$$\{\widehat{X}_r, X^m\} = D_r X^m, \tag{33}$$

where $\widehat{X}_r$ are fixed backgrounds and minimal immersions of the supermembrane. These backgrounds are responsible of the NC structure of the supermembrane and they are associated with the non-trivial central charges [5, 9]. In spite of the formal analogy between the structure of this theory and our model, there is an important difference. If we interpret the m transverse coordinates as the ones satisfying our NC G_2 structure and imposing $\lambda_r = \lambda$, the mass term contribution has its origin in a particular set of the structure constants f^C_{AB} of $SU(N)$ although in our case the mass terms are directly fixed by the NC G_2 structure.

In a type IIB superstring, a D-instanton matrix model for a massive SYM, without extra terms, has a fuzzy sphere and a fuzzy torus as possible solutions. In this case, the mass term is negative leading to some instabilities, although it is free to be set to different values. However, the origin of this extra term is not well understood [22]. In our model even if we find a matrix model that allows us to fix its mass term to our constant value, this mass coupling also remains unclear to us.

5. Conclusions

In this study, we have presented a NC G_2 structure extending results of NC Calabi–Yau manifolds. In this way, singularities of G_2 manifolds can be deformed by NC algebras. Using a algebraic orbifold method, we have given an explicit analysis for building a NC G_2 structure. In particular, we have developed and solved the non-commutativity parameter constraint equations defining such a deformed geometry. Then, we have shown that there are eight possible solutions having similar features of Yang–Baxter equations. Using a combinatorial argument, we have found eight matrix representations for such solutions.

Our results could be extended to Spin(7) holonomy manifold. The latter is an eight-dimensional manifold with Spin(7) holonomy group, being a subgroup of GL(8,**R**) which preserves a self-dual 4-form given by $\varphi = \mathrm{d}x_{1234} + \mathrm{d}x_{1256} + \mathrm{d}x_{1278} + \mathrm{d}x_{1357} - \mathrm{d}x_{1368} - \mathrm{d}x_{1458} - \mathrm{d}x_{1467} - \mathrm{d}x_{2358} - \mathrm{d}x_{2367} - \mathrm{d}x_{2457} + \mathrm{d}x_{2468} + \mathrm{d}x_{3456} + \mathrm{d}x_{3478} + \mathrm{d}x_{5678}$. The invariance of such a form, under a discrete group of Spin(7), leads to non-trivial constraint equations defining NC Spin(7) manifolds.

A matrix model of M-theory has been found for the commutative solution of the G_2 structure. It is a particular case of the deformed one. It satisfies trivially the solution to the vacuum equation of motion. However, this is not the situation for the NC geometries. We have shown that higher energy corrections are needed to satisfy such equations. These extra quantities being mass terms give information about NC structure. We argue that they appear as a consequence of the resolution of the singularity by introducing a NC algebra. The explicit coupling that leads to these mass terms in the regularized matrix model remains unclear to us. In order to find a complete connection with the matrix model for NC solutions, a more extensive analysis would be required.

On the other hand, a paper [26] dealing with topological transitions in fuzzy spaces has appeared recently. It also involves mass terms producing a topological change. Its suggested origin is a Yukawa interaction term in the D0-brane action. It has a certain resemblance to our case studied here, although we do not know if there could be an underlying relation between both approaches. We leave these open questions for future work.

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Appendix

In this appendix, we want to show that our eight representations satisfy equation (7). To do so, we should prove the following constraint equations:

$$M_i^2 = \mathbf{I} \tag{A.1}$$

$$M_i M_j = \Theta_{ij} M_j M_i \qquad i \neq j. \tag{A.2}$$

First, let us denote a nonzero coefficient $(a_i)_{mn}$, of the matrix M_i, by the position that occupy in rows and columns $(m, n)_{(s_i)_k}$, where $(s_i)_k$ its sign.

(i) $M_i^2 = \mathbf{I}$. It is easy to show this property. Indeed, from the construction of the matrices, we can see that there are no repeated coefficients and there is only one nonzero coefficient per row or column. Since the matrices are symmetric and traceless, the product between one term and its adjoint is the only contribution per line to the diagonal. For a given matrix M_i, the signs depend only on k which is the same for both type of terms. The product of signs is then the square of each sign. Since they are real square roots of the identity, we have

$$(M_i)^2_{mn} = (m,n)_{(s_i)_k}(n,m)_{(s_i)_k} = (m,m)_+ = \mathbf{I}. \tag{A.3}$$

(ii) $M_i M_j = \Theta_{ij} M_j M_i$ $i \neq j$. In order to prove this statement we need to check the following:

(1) The nonzero coefficients of $M_i M_j$ are the same to the ones of $M_j M_i$.
(2) The relative sign between each of the coefficients of the two products is $s_{ij} = \Theta_{ij} s_{ji}$.

In what follows, we suppose $i < j$ without any lack of generality. The matrix M_i can be expressed in terms of the nonzero coefficients as $\left\{(a_i)_{mn}, \left(a_i^{\dagger}\right)_{mn} x\right\}$ for $n > m$. The product of coefficients then takes the following form:

$$(a_i a_j)_{mo} = (a_i)_{m(m+2^{i-1})}(a_j)_{(m+2^{i-1})(m+2^{i-1}+2^{j-1}=o)}. \tag{A.4}$$

In an abbreviated notation, the product of the two matrices is given by

$$(M_i M_j) = \left\{a_i a_j, a_i a_j^{\dagger}, a_i^{\dagger} a_j, a_i^{\dagger} a_j^{\dagger}\right\}. \tag{A.5}$$

If this relation is proved for $a_i a_j$ and $a_i a_j^{\dagger}$, then their adjoint terms will also satisfy it. Let us first deal with $a_i a_j$. Indeed, the terms that contribute in the computations are

$$a_i a_j : (m,n)_{(s_i)_k}(n,o)_{(s_j)_l} = (m,o)_{s_{ij}} \tag{A.6}$$
$$a_j a_i : (m,r)_{(s_j)_l}(r,o)_{(s_i)_q} = (m,o)_{s_{ji}}. \tag{A.7}$$

(1) We will not be concerned about the signs.

Given $(a_i)_{mn}$ if there exists a $(a_j)_{no}$, then one can find $(a_j)_{mr}$ and $(a_i)_{ro}$ such that $(a_i)_{mn}(a_j)_{no} = (a_j)_{mr}(a_i)_{ro}$.[8]

(a) Moreover, we can find $(a_j)_{mr}$ once a nonzero coefficient exists in the line m, as there is just one per line or column. In this way, we have $(a_i)_{mn}$, $0 < m \leqslant 2^{i-1}$ and $0 < m < 2^{j-1}$ as required for any matrix labelled by j. Then, the term exists and by definition is given by $r = m + 2^{j-1}$.
(b) On the other hand, $(a_i)_{ro}$ exists if there is a coefficient in the line r of the matrix a_i satisfying $r = k + 2^i v'$ for some v'. This implies that $m = k + 2^i v_u$. Using (a), we have $r = k + 2^i v_x$ for $v_x = v_u + 2^{j-(1+i)}$, so the term also exists and by definition $o = r + 2^{i-1}$.

In fact, given $(a_i)_{mn}$, $(a_i)_{ro}$ is a coefficient with the same k translated in $2^{j-(i+1)}$ units of v. For instance, the product can be represented, as in [5], by

$$ij : m \stackrel{2^{i-1}}{\rightarrow} n \stackrel{2^{j-1}}{\rightarrow} o \tag{A.8}$$
$$ji : m \stackrel{2^{j-1}}{\rightarrow} r \stackrel{2^{i-1}}{\rightarrow} o. \tag{A.9}$$

Since (a) and (b) are verified, then we have

$$(a_i)_{mn}(a_j)_{no} = (a_j)_{mr}(a_i)_{ro}. \tag{A.10}$$

To prove (2) note that the relation between the signs is $s_{ij} = \Theta_{ij} s_{ji}$, being equivalent to verifying that

$$(s_i)_{mn}(s_j)_{no} = \Theta_{ij}(s_j)_{mr}(s_i)_{ro}. \tag{A.11}$$

[8] Only for a given $(a_i)_{mn}$ with $v = 0$ and $i \leqslant j$, it is guaranteed that $(a_j)_{no}$ exists.

For a given matrix, the signs depend only on k. This means that any value in j for a given k has the same sign. Then, we have

$$(s_i)_{mn} = (s_i)_{ro}. \tag{A.12}$$

It remains to describe $(s_j)_{no}$ in terms of $(s_j)_{mr}$. By definition $s_{k_i} = \prod_{\ell'} \Theta_{\ell' i}$, then we should find

$$\prod_{\ell'} \Theta_{\ell' i} = \Theta_{ij} \prod_{q'} \Theta_{q' j}. \tag{A.13}$$

However we do not need to know the explicit decomposition in terms of Θ. To obtain the relation between the two signs, it is enough to know their relative values of k. We will denote by k_n the value associated with the sign $(s_j)_{no}$ and, respectively, k_m to $(s_j)_{mr}$. Since $(a_i)_{mn}$ is given, the relative difference between n and m is known and we have

$$n = m + 2^{i-1} \tag{A.14}$$

$$k_n^j = 2^{i-1} k_m^j. \tag{A.15}$$

Since $j > i$, and from the definition of sign product, one can check that Θ_{ij} changes in s_j in each 2^{i-1} alternating values of k_j. So, we have the following:

$$(s_j)_{no} = \Theta_{ij}(s_j)_{mr}. \tag{A.16}$$

It follows that

$$s_{ij} = \Theta_{ij} s_{ji}. \tag{A.17}$$

The same results can be obtained for $\left(a_i a_j^\dagger\right)$ after making minor changes to the argument. Using (1) and (2), for any i and j, one gets

$$Y_i Y_j = \Theta_{ij} Y_j Y_i. \tag{A.18}$$

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